

RIGIDITY THEOREMS OF CONFORMAL CLASS ON COMPACT MANIFOLDS WITH BOUNDARY

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ABSTRACT. Let M be a compact manifold with boundary. In this paper, we discuss some rigidity theorems of metrics in a same conformal class that fixes the boundary and satisfy certain integral conditions on the the scalar curvatures and the mean curvatures on the boundary. No condition on the first eigenvalues of operators is need.

1. INTRODUCTION

Let (M, g_0) be a compact n -dimensional Riemannian smooth manifold with $n \geq 2$ and nonempty smooth boundary ∂M (possibly non connected). Let R_{g_0} denote the scalar curvature of (M, g_0) and let $H_{g_0} = \text{div}_{g_0} \eta_{g_0}$ denote the mean curvature of ∂M in (M, g_0) , in the direction of the exterior conormal $\eta = \eta_{g_0}$. If $n = 2$ then $K_{g_0} = R_{g_0}/2$ denotes the Gaussian curvature and $H_{g_0} = \kappa_{g_0}$ denotes the geodesic curvature of ∂M with respect to g_0 .

Recall the conformal class of a metric g on M , say $[g]$, is the set of metrics of the form $\tilde{g} = \mu g$, where μ is a positive smooth function defined on M . Escobar [3] dealt with the following question:

Given a metric $g \in [g_0]$ with $R_g = R_{g_0}$ in M , and $H_g = H_{g_0}$ on ∂M , when is $g = g_0$?

As Escobar observed in [4], this question does not have a positive answer in general. Indeed, he gave the description of the conformally flat metrics $g \in [\delta_{ij}]$ on the ball $B = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, with $n \geq 3$, having constant scalar curvature and constant mean curvature on ∂B . By this classification theorem, there is a non compact set of metrics $g \in [\delta_{ij}]$ with $R_g = 0$ and $H_g = 1$. Also in [4], Escobar showed that on the annulus $A_{a,b} = \{x \in \mathbb{R}^n \mid a < |x| < b\}$ there are several metrics $g \in [\delta_{ij}]$ with the same constant scalar curvature and the same constant mean curvature on the boundary, provided a/b is big enough.

In terms of PDE's, the scalar curvatures and mean curvatures of the metrics g_0 and $g \in [g_0]$ satisfy the following differential equations. For $n \geq 3$, we write $g = u^{\frac{4}{n-2}} g_0$. It holds:

$$(1) \quad \begin{cases} L_{g_0} u + c(n) R_{g_0} u^{\frac{n+2}{n-2}} = 0, & \text{in } M, \\ B_{g_0} u - 2c(n) u^{\frac{n}{n-2}} = 0, & \text{on } \partial M, \end{cases}$$

2000 *Mathematics Subject Classification.* Primary ; Secondary .
This work is partially supported by CNPq.

where $c(n) = \frac{n-2}{4(n-1)}$, $L_{g_0}(u) = \Delta_{g_0}u - c(n)R_{g_0}u$ and $B_{g_0}(u) = \frac{\partial u}{\partial \eta} + 2c(n)H_{g_0}u$. For $n = 2$, we write $g = e^{2u}g_0$. It holds:

$$(2) \quad \begin{cases} L_{g_0}u + K_g e^{2u} = 0, & \text{in } M, \\ B_{g_0}u - k_g e^u = 0, & \text{on } \partial M, \end{cases}$$

where $L_{g_0}(u) = \Delta_{g_0}u - K_{g_0}$ and $B_{g_0}(u) = \frac{\partial u}{\partial \eta} + \kappa_{g_0}$. The second pair of operators Escobar talked about are defined by

$$(3) \quad \begin{cases} \mathcal{L}_{g_0} = \Delta_{g_0} + \frac{1}{n-1}R_{g_0}, & \text{in } M, \\ \mathcal{B}_{g_0} = \frac{\partial}{\partial \eta_{g_0}} - \frac{1}{n-1}H_{g_0}, & \text{on } \partial M. \end{cases}$$

The operators $(\mathcal{L}_{g_0}, \mathcal{B}_{g_0})$ are the linearizations of (1), when $n \geq 3$, at $u = 1$, and (2), when $n = 2$, at $u = 0$, for the cases $R_g = R_{g_0}$ and $H_g = H_{g_0}$. Escobar [3] proved the following

Theorem A (Theorem 1 of [3]). *Let $g \in [g_0]$ with $R_g = R_{g_0}$ and $H_g = H_{g_0} \leq 0$. If both first eigenvalues $\lambda_1(\mathcal{L}_g, \mathcal{B}_g)$ and $\lambda_1(\mathcal{L}_{g_0}, \mathcal{B}_{g_0})$ are positive or one of them is equal to zero then $g = g_0$.*

As a consequence, he obtained the following

Corollary A (Corollary 2 of [3]). *Let $g \in [g_0]$ satisfying $R_g = R_{g_0} \leq 0$ and $H_g = H_{g_0} \leq 0$. Then $g = g_0$.*

Min-Oo [8] conjectured the following: *Let (M, g) be an n -dimensional compact Riemannian manifold with boundary and scalar curvature $R_g \geq n(n-1)$. Assume the boundary is isometric to the standard sphere S^{n-1} and is totally geodesic in M . Then, (M, g) is isometric to the upper hemisphere S_+^n .* Min-Oo's conjecture fell in 2011, when Brendle, Marques and Neves exhibited a beautiful counterexample. On the other hand, Hang and Wang [6] proved Min-Oo's conjecture is true for the case the metric is conformal to the metric of S_+^n . They proved the following

Theorem B (Theorem 3.4 of [6]). *Let $g \in [g_{S^n}]$ on S_+^n . Assume that the scalar curvature $R_g \geq R_{g_{S^n}} = n(n-1)$ and $g = g_{S^n}$ on the boundary ∂S_+^n . Then $g = g_{S^n}$.*

Based on Theorem B and Min-Oo's Problem, we are interested into the following question:

Given a metric $g \in [g_0]$ with $g = g_0$ on ∂M and $R_g \geq R_{g_0}$ in M , when is $g = g_0$?

The upper hemisphere S_+^n is a static manifold, that means there is a smooth function f satisfying the equation

$$(4) \quad \begin{cases} f\text{Ric} - \nabla^2 f + (\Delta f)g = 0, & \text{in } M \setminus \partial M \\ f > 0 \text{ in } M \setminus \partial M, \text{ and } f = 0, & \text{on } \partial M \end{cases}$$

As a solution of (4) for S_+^n , we take, for instance, the height function $f(x) = x_{n+1}$, for all $x = (x_1, \dots, x_{n+1}) \in S_+^n$. By taking the trace in (4), we see static manifolds are solutions of $\mathcal{L}_g f = 0$, for some $f \in C^2(M)$ that is positive in M and vanishes on ∂M .

Our first theorem is

Theorem 1. *Let $g = \mu^2 g_0$ a metric in the class $[g_0]$ such that $g = g_0$ on ∂M . Let $f \in C^1(M) \cap C^2(M \setminus \partial M)$, positive a.e. such that*

$$(5) \quad \int_M f(R_g - R_{g_0}) d\text{vol}_{g_0} + 2 \int_{\partial M} f(H_g - H_{g_0}) d\mathcal{H}_{g_0}^{n-1} \geq 0.$$

If $\int_M \mathcal{L}_{g_0} f (1 - \mu^{-2}) d\text{vol}_{g_0} \geq 0$ then $g = g_0$.

A direct application of Theorem 1, by using $\mathcal{L}_{g_0}(1) = \frac{1}{n-1} R_{g_0}$, is

Corollary 2. *Let $g = \mu^2 g_0$ be a metric in the conformal class $[g_0]$ such that $g = g_0$ on ∂M . Assume that*

$$(6) \quad \int_M (R_g - R_{g_0}) d\text{vol}_{g_0} + 2 \int_{\partial M} (H_g - H_{g_0}) d\mathcal{H}_{g_0}^{n-1} \geq 0.$$

If $\int_M R_{g_0} (1 - \mu^{-2}) d\text{vol}_{g_0} \geq 0$ then $g = g_0$.

For static metrics, it holds $\mathcal{L}_{g_0} f = 0$, for some $f \in C^2(M)$ that is positive in $M \setminus \partial M$ and vanishes along ∂M . Thus, we have

Corollary 3. *Let g_0 be a static metric on M and $g \in [g_0]$ such that $g = g_0$ on ∂M . If $R \geq R_0$ then $g = g_0$.*

Araujo [1] studied the functional

$$(7) \quad F(g) = \int_M R_g d\text{vol}_g + 2 \int_{\partial M} H_g d\sigma_g.$$

restricted to the subset of metrics $\mathcal{M}_{ab} = \{g \mid a \text{vol}_g(M) + b\mathcal{A}_g(\partial M) = 1\}$. He proved the critical points of F are the Einstein metrics with umbilical boundary that satisfy $b(n-1)R_g = 2naH_g$. It is worthwhile to point out assumption (6) of Corollary 2 does not imply $F(g) \geq F(g_0)$, since volume and area elements in (6) does not vary with the metric. Using Gauss-Bonnet Theorem, Corollary 2 in dimension 2 can be written as

Corollary 4. *Let (M, g_0) be a Riemannian surface with smooth boundary. Let $u \in C^2(M)$ with $u = 0$ on ∂M and consider the metric $g = e^{2u} g_0$. Assume*

$$\int_M K_g d\text{vol}_{g_0} + \int_{\partial M} \kappa_g d\mathcal{H}_{g_0}^1 \geq 2\pi\chi(M),$$

If $\int_M K_{g_0} (1 - e^{-2u}) d\text{vol}_{g_0} \geq 0$ then $u = 0$ in M .

Theorem 1 requires no condition on the first eigenvalue of \mathcal{L}_{g_0} (and even \mathcal{B}_{g_0}). However, the first eigenvalue

$$\lambda_1(\mathcal{L}_{g_0}) = \inf \left\{ \int_M (|\nabla \varphi|^2 - \frac{R_{g_0}}{n-1} \varphi^2) d\text{vol}_{g_0} \mid \varphi \in C_0^\infty(M), \int_M \varphi^2 d\text{vol}_{g_0} = 1 \right\}$$

satisfies $\mathcal{L}_{g_0} f + \lambda_1(\mathcal{L}_{g_0}) f = 0$, for some C^2 eigenfunction f that is positive in M and vanishes on ∂M . Thus, Corollary 5 below is a generalization of Corollary 3.

Corollary 5. *Let $g \in [g_0]$ satisfying $g = g_0$, on ∂M , and $R_g \geq R_{g_0}$. Assume $\lambda_1(\mathcal{L}_{g_0})(g - g_0) \leq 0$. Then $g = g_0$.*

Llarul [9], confirming a conjecture due Gromov, proved the following result: *If g is any metric on the sphere S^n satisfying $g \geq g_0$ and $R_g \geq R_{g_{S^n}} = n(n-1)$ then $g = g_{S^n}$. Furthermore, for domains $\Omega \subset S^n$, Hang and Wang [7] proved the following*

Theorem C (Proposition 1 of [7]). *Let Ω be a smooth domain in S_+^n and let $g \in [g_{S^n}]$ in $\bar{\Omega}$, satisfying $R_g \geq n(n-1)$ and $g = g_{S^n}$ on $\partial\Omega$. Then either $g = g_{S_+^n}$, in Ω , or $g > g_{S_+^n}$ and $H < H_{g_{S_+^n}}$.*

In this paper we prove

Theorem 6. *Assume $R_{g_0} \geq 0$ and $\mathcal{L}_{g_0}f \leq 0$, for some $f \in C^2(M \setminus \partial M) \cap C^1(M)$ positive a.e.. Let Ω be a smooth domain in M and let $g = \mu^2 g_0$, where $\mu \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is positive with $\mu|_{\partial\Omega} = 1$. Assume that $\chi_{\{\mu < 1\}} R_g \geq \chi_{\{\mu < 1\}} R_{g_0}$. Then, it holds*

$$(8) \quad g \geq g_0 \text{ in } \Omega, \text{ and } H_g \leq H_{g_0} \text{ in } \partial\Omega.$$

In addition, if $R_g \geq R_{g_0}$ then the inequalities in (8) are strict, unless $g = g_0$.

If g_0 is a static metric on a manifold with boundary then it holds

Corollary 7. *Let g_0 be a static metric on M with $R_{g_0} \geq 0$. Let $\Omega \subset M$ be a smooth domain and let $g = \mu^2 g_0$, where $\mu \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is positive with $\mu|_{\partial\Omega} = 1$. Assume that $\chi_{\{\mu < 1\}} R_g \geq \chi_{\{\mu < 1\}} R_{g_0}$. Then, it holds*

$$g \geq g_0 \text{ in } \Omega, \text{ and } H_g \leq H_{g_0} \text{ in } \partial\Omega.$$

In addition, if $R_g \geq R_{g_0}$ then the inequalities above are strict, unless $g = g_0$.

As another application, by Theorem 6 with $f = 1$, we obtain

Corollary 8. *Let g_0 be a metric on M with $R_{g_0} = 0$. Let $\Omega \subset M$ be a smooth domain. Consider the metric $g = \mu^2 g_0$, where $\mu \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is positive with $\mu|_{\partial\Omega} = 1$. Assume $\chi_{\{\mu < 1\}} R_g \geq 0$. Then, it holds*

$$g \geq g_0 \text{ in } \Omega, \text{ and } H_g \leq H_{g_0} \text{ in } \partial\Omega.$$

In addition, if $R_g \geq 0$ in Ω , then the inequalities above are strict, unless $g = g_0$.

ACKNOWLEDGEMENT

The authors are very grateful to Professor Andre Neves for your suggestions and comments, and Imperial College London for the kind hospitality. During this work, E. Barbosa and H. Mirandola were supported by a CNPq Postdoctoral Fellowship, and F. Vitorio was supported by a CNPq Universal Grant.

2. PROOF OF THEOREM 1

First, consider the case $n = 2$ and write $g = e^{2u} g_0$, with $u \in C^2(M \setminus \partial M) \cap C^1(M)$. Since $g = g_0$ on ∂M , one has $u|_{\partial M} = 0$. The geodesic curvatures κ_g, κ_{g_0} satisfy

$$(9) \quad \frac{\partial u}{\partial \eta} = \kappa_g e^u - \kappa_{g_0} = \kappa_g - \kappa_{g_0}, \text{ on } \partial M,$$

where $\eta = \eta_{g_0}$ is the outward unit normal vector of $(\partial M, g_0)$. Furthermore, the Gaussian curvatures K_g, K_{g_0} of (M, g_0) and (M, g) , respectively, satisfy

$$(10) \quad \Delta_{g_0} u - K_{g_0} + K_g e^{2u} = 0, \text{ in } M.$$

Using that $u|_{\partial M} = 0$, by (9) and integration by parts, we obtain

$$\begin{aligned}
\int_M e^{-2u} \Delta_{g_0} f &= \int_M f \Delta_{g_0} (e^{-2u}) + \int_{\partial M} (e^{-2u} \frac{\partial f}{\partial \eta} - f \frac{\partial (e^{-2u})}{\partial \eta}) \\
&= \int_M f \Delta_{g_0} (e^{-2u}) + \int_{\partial M} \frac{\partial f}{\partial \eta} + 2 \int_{\partial M} f \frac{\partial u}{\partial \eta} \\
&= \int_M [-2f e^{-2u} (\Delta_{g_0} u - 2|Du|_{g_0}^2) + \Delta_{g_0} f] + 2 \int_{\partial M} f (\kappa_g - \kappa_{g_0}) \\
&= \int_M [-2f e^{-2u} (K_{g_0} - K_g e^{2u} - 2|Du|_{g_0}^2) + \Delta_{g_0} f] \\
&\quad + \int_{\partial M} 2f (\kappa_g - \kappa_{g_0}).
\end{aligned}$$

Thus, since $\Delta_{g_0} f = \mathcal{L}_{g_0} f - 2K_{g_0} f$, we obtain

$$\begin{aligned}
\int_M e^{-2u} (\Delta_{g_0} f + 2K_{g_0} f) &= \int_M (\mathcal{L}_{g_0} f + 4f e^{-2u} |Du|_{g_0}^2) \\
&\quad + \int_M 2f (K_g - K_{g_0}) + \int_{\partial M} 2f (\kappa_g - \kappa_{g_0}).
\end{aligned}$$

Hence,

$$\int_M \mathcal{L}_{g_0} f (e^{-2u} - 1) = \int_M 4f e^{-2u} |Du|_{g_0}^2 + \int_M 2f (K_g - K_{g_0}) + \int_{\partial M} 2f (\kappa_g - \kappa_{g_0}).$$

By hypothesis, $\int_M \mathcal{L}_{g_0} f (1 - e^{-2u}) \geq 0$ and $\int_M 2f (K_g - K_{g_0}) + \int_{\partial M} 2f (\kappa_g - \kappa_{g_0}) \geq 0$. Hence, $Du = 0$, which together with the fact that $u|_{\partial M} = 0$, imply that $g = g_0$.

Now, we assume $n \geq 3$ and write $g = u^{\frac{4}{n-2}} g_0$, for some $u \in C^2(M \setminus \partial M) \cap C^1(M)$, positive in M and with $u = 1$ on ∂M . The mean curvatures $H_{g_0} = \operatorname{div}_{g_0} \eta_{g_0}$ and $H_g = \operatorname{div}_g \eta_g$ satisfy

$$(11) \quad \frac{\partial u}{\partial \eta} = \frac{n-2}{2(n-1)} (H_g u^{\frac{n}{n-2}} - H_{g_0}) = \frac{n-2}{2(n-1)} (H_g - H_{g_0}), \text{ on } \partial M,$$

where $\eta = \eta_{g_0}$. Furthermore, the scalar curvatures R_g and R_{g_0} satisfy

$$(12) \quad \Delta_{g_0} u - \frac{n-2}{4(n-1)} R_{g_0} u + \frac{n-2}{4(n-1)} R_g u^{\frac{n+2}{n-2}} = 0, \text{ in } M.$$

Let λ be a constant to be chosen later. Using that $u|_{\partial M} = 1$, integrating by parts

we obtain

$$\begin{aligned}
\int_M u^\lambda \Delta_{g_0} f &= \int_M f \Delta_{g_0} u^\lambda + \int_{\partial M} (u^\lambda \frac{\partial f}{\partial \eta} - f \frac{\partial(u^\lambda)}{\partial \eta}) \\
&= \int_M (f \lambda u^{\lambda-1} \Delta_{g_0} u + f \lambda (\lambda-1) u^{\lambda-2} |Du|_{g_0}^2) + \int_{\partial M} \frac{\partial f}{\partial \eta} - \lambda f \frac{\partial u}{\partial \eta} \\
&= \int_M [(f \lambda u^{\lambda-1} \Delta_{g_0} u + f \lambda (\lambda-1) u^{\lambda-2} |Du|_{g_0}^2) + \Delta_{g_0} f] \\
&\quad - \frac{(n-2)}{2(n-1)} \lambda \int_{\partial M} f (H_g - H_{g_0}) \\
&= \int_M [f \lambda u^{\lambda-1} (\frac{n-2}{4(n-1)} (R_{g_0} u - R_g u^{\frac{n+2}{n-2}}) + f \lambda (\lambda-1) u^{\lambda-2} |Du|_{g_0}^2)] \\
&\quad + \int_M \Delta_{g_0} f - \frac{(n-2)}{2(n-1)} \lambda \int_{\partial M} f (H_g - H_{g_0}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_M u^\lambda (\Delta_{g_0} f - \lambda f \frac{n-2}{4(n-1)} R_{g_0}) &= -\lambda \frac{n-2}{4(n-1)} \int_M f R_g u^{\lambda-1 + \frac{n+2}{n-2}} \\
&\quad + \int_M \Delta_{g_0} f + \lambda (\lambda-1) \int_M f u^{\lambda-2} |Du|_{g_0}^2 \\
&\quad - \frac{(n-2)}{2(n-1)} \lambda \int_{\partial M} f (H_g - H_{g_0})
\end{aligned}$$

We choose $\lambda = 1 - \frac{n+2}{n-2} = \frac{-4}{n-2}$. We obtain

$$\begin{aligned}
\int_M u^{\frac{-4}{n-2}} \mathcal{L}_{g_0} f &= \int_M f \frac{R_g}{n-1} + \int_M (\mathcal{L}_{g_0} f - \frac{R_{g_0}}{n-1} f) \\
&\quad + \frac{4(n+2)}{(n-2)^2} \int_M f u^{\frac{-2n}{n-2}} |Du|_{g_0}^2 + \frac{2}{n-1} \int_{\partial M} f (H_g - H_{g_0}).
\end{aligned}$$

We obtain

$$\begin{aligned}
\int_M \mathcal{L}_{g_0} f (u^{\frac{-4}{n-2}} - 1) &= \frac{4(n+2)}{(n-2)^2} \int_M f u^{\frac{-2n}{n-2}} |Du|_{g_0}^2 \\
&\quad + \frac{1}{n-1} \int_M f (R_g - R_{g_0}) + \frac{2}{n-1} \int_{\partial M} f (H_g - H_{g_0}).
\end{aligned}$$

By hypothesis, $g = u^{\frac{4}{n-2}} g_0$ satisfies $\int_M \mathcal{L}_{g_0} f (1 - u^{\frac{-4}{n-2}}) \geq 0$. Using (5), we obtain that $Du = 0$. Since $u|_{\partial M} = 1$ one has $u = 1$ in M ; hence $g = g_0$. Theorem 1 is proved.

3. PROOF OF THEOREM 6

First, consider the case $n = 2$ and write $g = e^{2u} g_0$ with $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Since $u|_{\partial\Omega} = 0$ the geodesic curvatures κ_g, κ_{g_0} satisfy

$$(13) \quad \frac{\partial u}{\partial \eta} = \kappa_g e^u - \kappa_{g_0} = \kappa_g - \kappa_{g_0}, \text{ in } \partial\Omega,$$

where $\eta = \eta_{g_0}$ is the outward unit normal vector of $(\partial\Omega, g_0)$. Furthermore, the Gaussian curvatures K_g, K_{g_0} of (Ω, g_0) and (Ω, g) , respectively, satisfy

$$(14) \quad \Delta_{g_0} u = K_{g_0} - K_g e^{2u}.$$

Let $\bar{u} = \min\{u, 0\}$. It turns that \bar{u} is continuous and, in the sense of distributions, it holds $\Delta_{g_0} \bar{u} \leq \chi_{\{u < 0\}} \Delta_{g_0} u = \chi_{\{u < 0\}} (K_{g_0} - K_g e^{2u}) \leq \chi_{\{u < 0\}} K_{g_0} (1 - e^{2u})$, since $\chi_{\{u < 0\}} K_g \geq \chi_{\{u < 0\}} K_{g_0}$. Let $A_u = \chi_{\{u < 0\}} K_{g_0} (1 - e^{2u})$. We have that $A_u = A_{\bar{u}}$ is a nonnegative continuous function, and

$$\Delta_{g_0} \bar{u} \leq A_{\bar{u}}, \text{ in } M,$$

in the sense of distributions. The function $A_{\bar{u}}$ is Lipschitz in $\bar{\Omega}$. In fact, given $x, x_0 \in \bar{\Omega}$, if either $x, x_0 \in \{u < 1\}$, or $x, x_0 \in \{u \geq 1\}$, we have $|A_{\bar{u}}(x) - A_{\bar{u}}(x_0)| = \chi_{\{u < 0\}} |K_0(x) e^{2u(x)} - K_0(x_0) e^{2u(x_0)}| \leq M d_{g_0}(x, x_0)$, for some $M > 0$, since $K_0 e^{2u} \in C^1(\bar{\Omega})$. Thus, we assume that $u(x) < 1$ and $u(x_0) \geq 1$. In this case, $|A_{\bar{u}}(x) - A_{\bar{u}}(x_0)| = |A_{\bar{u}}(x)| = |K_0(x)| (1 - e^{2u}) \leq (\max |K_0|) (e^{2u(x_0)} - e^{2u(x)}) \leq M d_{g_0}(x, x_0)$, for some $M > 0$, since $u \in C^1(\bar{\Omega})$.

Now, consider $\bar{v} : \bar{\Omega} \rightarrow \mathbb{R}$ a solution of the Dirichlet problem

$$\Delta_{g_0} \bar{v} = A_{\bar{u}}, \text{ in } \Omega, \text{ and } \bar{v}|_{\partial\Omega} = 0.$$

Since $A_{\bar{u}}$ is Lipschitz in $\bar{\Omega}$ we have $v \in C^2(\bar{\Omega})$ (see Theorem 8.34, pg 211, of [5]). Furthermore, since $\Delta_{g_0}(\bar{u} - \bar{v}) \leq 0$ and $(\bar{u} - \bar{v})|_{\partial\Omega} = 0$, one has $\bar{v} \leq \bar{u} \leq 0$. This implies $1 - e^{2\bar{u}} \leq 1 - e^{2\bar{v}}$ and $\chi_{\{\bar{u} < 0\}} \leq \chi_{\{\bar{v} < 0\}}$, hence $A_{\bar{u}} \leq A_{\bar{v}}$ in $\bar{\Omega}$, since $K_{g_0} \geq 0$. Thus,

$$\Delta_{g_0} \bar{v} \leq A_{\bar{v}}, \text{ in } \Omega, \text{ and } \bar{v}|_{\partial\Omega} = 0.$$

Let v be defined by

$$v(x) = \begin{cases} \bar{v}(x), & \text{if } x \in \bar{\Omega}; \\ 0, & \text{if } x \in M \setminus \bar{\Omega}. \end{cases}$$

We have A_v is Lipschitz and v and satisfies

$$\Delta_{g_0} v \leq A_v, \text{ in } M,$$

in the sense of distributions, and $v|_{\partial M} = 0$. Let $\omega \in C^2(M)$ be a solution of

$$\Delta_{g_0} \omega = A_v, \text{ in } M, \text{ and } \omega|_{\partial M} = 0.$$

Since Ω is a domain in M , it follows $v = 0$ in a neighborhood \mathcal{U} of ∂M in M , hence $A_v = 0$ in \mathcal{U} , hence $w \in C^2(M)$. Furthermore, we have $\Delta_{g_0}(v - w) \leq 0$, $(v - w)|_{\partial M} = 0$ and $v = 0$ in \mathcal{U} . These imply $w \leq v \leq 0$ and

$$(15) \quad \frac{\partial \omega}{\partial \eta} \geq \frac{\partial v}{\partial \eta} = 0 \text{ on } \partial M.$$

In addition, we also have $A_v \leq A_\omega$. Hence, $\Delta_{g_0} \omega \leq A_\omega$. Thus, the metric $\tilde{g} = e^{2\omega} g_0$ satisfies

$$\begin{aligned} K_{\tilde{g}} &= e^{-2\omega} (K_{g_0} - \Delta_{g_0} \omega) \geq e^{-2\omega} (K_{g_0} - A_\omega (1 - e^{2\omega})) \\ &\geq e^{-2\omega} (K_{g_0} - \chi_{\{\omega < 0\}} K_{g_0} (1 - e^{2\omega})) \\ &= e^{-2\omega} (K_{g_0} (1 - \chi_{\{\omega < 0\}}) + \chi_{\{\omega < 0\}} K_{g_0} e^{2\omega}) \\ &= K_{g_0}. \end{aligned}$$

The last equality follows just analyzing the cases $\omega < 0$ and $\omega = 0$. And, by (15), one has $k_{\tilde{g}} = \frac{\partial \omega}{\partial \eta} + k_{g_0} \geq k_{g_0}$.

Since $\omega \leq 0$ and $\mathcal{L}_{g_0} f \leq 0$, for some $f \in C^2(M \setminus \partial M) \cap C^1(M)$ positive a.e., we obtain $\mathcal{L}_{g_0} f (1 - e^{-2\omega}) \geq 0$. By Theorem 1, one has $\tilde{g} = g_0$, hence $\omega = 0$. This implies $v = \bar{v} = \bar{u} = 0$, hence $u \geq 0$. We obtain $g \geq g_0$. Moreover, using $u \geq 0$ and $u|_{\partial\Omega} = 0$, by (9), one has $\frac{\partial u}{\partial \eta} \leq 0$, hence $\kappa_g \leq \kappa_{g_0}$.

Now, assume further $K_g \geq K_{g_0}$ in Ω . Using (14), one has $\Delta_{g_0} u \leq 0$. Since $\frac{\partial u}{\partial \eta} \leq 0$ and $u|_{\partial\Omega} = 0$, by interior maximum principle and Hopf Lemma, it follows $u = 0$ in $\bar{\Omega}$, provided $u = 0$ somewhere in Ω or $\frac{\partial u}{\partial \eta} = 0$ somewhere on $\partial\Omega$.

Now, consider the case $n \geq 3$ and write $g = u^{\frac{4}{n-2}} g_0$, for some positive function $u \in C^2(\bar{\Omega})$ with $u = 1$ on $\partial\Omega$. The function $\bar{u} = \min\{1, u\}$ is continuous in $\bar{\Omega}$ and satisfies $\bar{u}|_{\partial\Omega} = 1$. Furthermore, $\Delta_{g_0} \bar{u} \leq \chi_{\{u < 1\}} \Delta_{g_0} u$, in the sense of distributions. Thus, using $\chi_{\{u < 1\}} R_g \geq \chi_{\{u < 1\}} R_{g_0} \geq 0$, by (12), we obtain

$$\begin{aligned} (16) \quad \Delta_{g_0} \bar{u} &\leq \frac{n-2}{4(n-1)} \chi_{\{u < 1\}} (R_{g_0} u - R_g u^{\frac{n+2}{n-2}}) \\ &\leq \frac{n-2}{4(n-1)} \chi_{\{\bar{u} < 1\}} R_{g_0} (\bar{u} - \bar{u}^{\frac{n+2}{n-2}}) \\ &= A_{\bar{u}} \bar{u}, \text{ in } M, \end{aligned}$$

in the sense of distributions, where $A_{\bar{u}} = \frac{n-2}{4(n-1)} \chi_{\{\bar{u} < 1\}} R_{g_0} (1 - \bar{u}^{\frac{4}{n-2}})$. Note that $A_{\bar{u}} \geq 0$ is Lipschitz in $\bar{\Omega}$.

Let $\bar{v} \in C^2(\bar{\Omega})$ be a solution of the Dirichlet problem

$$\Delta_{g_0} \bar{v} - A_{\bar{u}} \bar{v} = 0 \quad \text{and} \quad \bar{v}|_{\partial\Omega} = 1$$

(see Theorem 8.34, pg 211, of [5]). Since $\Delta_{g_0}(\bar{v} - \bar{u}) - A_{\bar{u}}(\bar{v} - \bar{u}) \geq 0$ and $(\bar{u} - \bar{v})|_{\partial\Omega} = 0$, we have $\bar{v} \leq \bar{u}$, since, by the strong maximum principle, $\bar{v} - \bar{u}$ cannot achieve a nonnegative maximum in the interior of Ω , unless $\bar{u} = \bar{v}$. We obtain $\chi_{\{\bar{v} < 1\}} \geq \chi_{\{\bar{u} < 1\}}$ and $1 - \bar{v}^{\frac{4}{n-2}} \geq 1 - \bar{u}^{\frac{4}{n-2}}$. This implies $A_{\bar{v}} \geq A_{\bar{u}}$. Hence,

$$\Delta_{g_0} \bar{v} - A_{\bar{v}} \bar{v} \leq 0, \text{ in } M, \quad \text{and} \quad \bar{v}|_{\partial\Omega} = 1.$$

Let $v : M \rightarrow \mathbb{R}$ be defined by

$$v(x) = \begin{cases} \bar{v}(x), & \text{if } x \in \bar{\Omega}; \\ 1, & \text{if } x \in M \setminus \bar{\Omega}. \end{cases}$$

Note that $v \leq 1$ in M and A_v is Lipschitz. Furthermore, it holds

$$(17) \quad \Delta_{g_0} v - A_v v \leq 0, \text{ in } M,$$

Let $w \in C^2(M)$ be a solution of the Dirichlet problem

$$(18) \quad \Delta_{g_0} w - A_v w = 0, \text{ in } M, \text{ and } w|_{\partial M} = 1.$$

Since $A_v \geq 0$, by the strong maximum principle, $-w$ cannot achieve a nonnegative maximum in $M \setminus \partial M$, unless w is constant. Hence $w > 0$, since $w|_{\partial M} = 1$. Furthermore, by (17) and (18), we have $\Delta_{g_0}(w - v) - A_v(w - v) \geq 0$, in M , in the sense of distributions, and $w - v = 0$ in ∂M . Again by the strong maximum principle, we obtain $w \leq v \leq 1$ in M , hence $A_w \geq A_v$. Thus, by (18),

$$(19) \quad \Delta_{g_0} w - A_w w \leq 0, \text{ in } M, \text{ and } w|_{\partial M} = 1.$$

Consider the metric $\tilde{g} = w^{\frac{4}{n-2}} g_0$. By (12), the scalar curvatures $R_{\tilde{g}}$ and R_{g_0} satisfy

$$\begin{aligned} R_{\tilde{g}} &= w^{-\frac{n+2}{n-2}} (R_{g_0} w - \frac{4(n-1)}{n-2} \Delta_{g_0} w) \\ &\geq w^{-\frac{n+2}{n-2}} (R_{g_0} w - \frac{4(n-1)}{n-2} A_w w) \\ &= w^{-\frac{n+2}{n-2}} ((1 - \chi_{\{w < 1\}}) R_{g_0} w + \chi_{\{w < 1\}} R_{g_0} w^{\frac{n+2}{n-2}}) \\ &= R_{g_0}. \end{aligned}$$

The last equality follows just by analyzing the cases $w < 1$ and $w = 1$. Furthermore, since $w \leq 1$ and $w|_{\partial M} = 1$, we have $\frac{\partial w}{\partial \eta} \geq 0$ on ∂M . By (11), the mean curvatures H_{g_0} and $H_{\tilde{g}}$ satisfy

$$H_{\tilde{g}} = H_{g_0} + \frac{2(n-1)}{n-2} \frac{\partial w}{\partial \eta} \geq H_{g_0}.$$

Since $1 - w^{\frac{-4}{n-2}} \leq 0$, and there exists $f \in C^2(M \setminus \partial M) \cap C^1(M)$ positive a.e. such that $\mathcal{L}_{g_0} f \leq 0$, we have $\mathcal{L}_{g_0} f(1 - w^{\frac{-4}{n-2}}) \geq 0$. By Theorem 1, it holds $\tilde{g} = g_0$, hence $w = 1$. Thus $v = \bar{v} = \bar{u} = 1$, hence $u \geq 1$. This implies $g \geq g_0$. Moreover, since $u \geq 1$ and $u|_{\partial M} = 1$, we also have $\frac{\partial u}{\partial \eta} \leq 0$, hence $H_g \leq H_{g_0}$.

Now, we assume further $R_g \geq R_{g_0}$ in M . Since $u \geq 1$, by (12), one has $\Delta_{g_0} u \geq 0$. Thus, if $u = 1$, somewhere in Ω , or $\frac{\partial u}{\partial \eta} = 0$, somewhere in $\partial\Omega$, then, by interior maximum principle or Hopf Lemma, it holds $u = 1$ in M . Theorem 6 is proved.

REFERENCES

- [1] Araujo, H., *Critical Points of the Total Scalar Curvature plus Total Mean Curvature Functional*. Indiana Math. J. 52 (2003), 85 – 107.
- [2] Brendle, S.; Marques, F. C.; Neves, A.: *Deformations of the hemisphere that increase scalar curvature*. Invent. Math. 185 (2011), 175 – 197.
- [3] Escobar, J. F., *Uniqueness and non-uniqueness of metrics with prescribed scalar and mean curvature on compact manifolds with boundary*, J. Funct. Anal. 202 (2003), no. 2, 424 – 442.
- [4] Escobar, J. F., *Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate*, Comm. Pure Appl. Math. 43 (1990) 857 – 883.
- [5] Gilbarg, D. and Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order*. Classic in Mathematics, ISSN 1431-0821, 2001.
- [6] Hang, F., Wang, X., *Rigidity and non-rigidity results on the sphere*. Commun. Anal. Geom. 14 (2006), 91 – 106.
- [7] Hang, F., Wang, X., *Rigidity Theorems for Compact Manifolds with Boundary and Positive Ricci Curvature*, J. Geom Anal 19 (2009), 628 – 642.
- [8] Min-Oo, M. *Scalar curvature rigidity of the hemisphere*. Unpublished, 1995.
- [9] Llarull, M. *Sharp estimates and the Dirac operator*. Math. Ann. 310 (1998), 55 – 71.

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